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NBW-003-016403

Seat No. _______

M. Sc. (Sem. IV) (CBCS) Examination

April / May - 2017

Mathematics : 4003

(Number Theory - II)

Faculty Code : 003

Subject Code : 016403

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions :

(1) There are five questions in this paper

(2) Each question carries 14 marks

(3) All questions are compulsory

(4) Figures to the right indicate full marks.

1 Fill in the blanks : (Each question carries two marks)

(a) If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions in the n$^{\text{th}}$ row and $\frac{a}{b}$ is less than $\frac{c}{d}$ then ...... and $\frac{c}{d}$ are consecutive Farey fractions in the $(n+1)^{\text{th}}$ row.

(b) If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions in the n^{th} row then $\left|\frac{a}{b}-\frac{a+c}{b+d}\right| \le$ ....

(c) There is a polynomial $f_n(x)$ with degree n, leading coefficient 1 and integer coefficients such that $f_n(2\cos\theta) =$ ...... for every $n \ge 1$ and real θ.

(d) If θ is an irrational number and $\frac{a}{b}$ is a rational number such that $b > 0$ and $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$ then $\frac{a}{b}$ = ... for some n.

(e) If continued fraction expansion of an irrational θ is purely periodic then $\frac{1}{\theta}$ lies between _______ and _______.

(f) If θ is an irrational, $\frac{a}{b}$ is a rational number such that $|\theta b - a| < |\theta k_n - h_n|$ for some $n \geq 1$ then $b \geq$ ______.

(g) The Diophantine equation $ax + by = c$ has a solution if and only if _________ divides c.

2 Attempt any two of the following :

(a) If θ is an irrational number then prove that there are infinitely many rational numbers $\frac{a}{b}$ such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{b^2}$. **7**

(b) Prove that π is irrational using elementary method. **7**

(c) Suppose $f(x) = a_n X^n + a_{n-1} X^{n-1} + \ldots\ldots + a_0$ is a polynomial of degree n, with integer coefficients and suppose $\frac{a}{b}$ is a rational number with $b > 0$, $(a, b) = 1$ and $\frac{a}{b}$ is a root of $f(x)$. Prove that b divides a_n and a divides a_0. Deduce that if $a_n = 1$ then $\frac{a}{b}$ must be an integer and also deduce that if an integer x has a rational n^{th} root then it must be an integer. **7**

3 All are compulsory :

(a) If θ is a quadratic irrational such that (i) $\theta > 1$ (ii) $-1 < \theta' < 0$ then prove that continued fraction expansion of θ is purely periodic. **6**

(b) If θ is irrational and $\theta = \langle a_0, a_1, \ldots a_n, \ldots \rangle$ then prove that $k_n < \theta_n k_{n-1} + k_{n-2} < k_{n+1}$ for all $n \geq 0$. **4**

(c) Prove that $x^2 = y^3 + 7$ has no solution in integers. **4**

OR

3 All are compulsory :

(a) Suppose θ is irrational and $\frac{a}{b}$ is a rational number such that $\left|\theta - \frac{a}{b}\right| < \frac{1}{2b^2}$ then prove that $\frac{a}{b}$ is the n^{th} convergent for some n. **6**

(b) Find general solutions (if any) of the following Diophantine equations : **4**

(i) $2x + 5y = 11$

(ii) $100x + 101y = 2017$

(c) Find the value of following continued fractions : **4**

(i) $\langle 1, 3, 3, 3, 3, \ldots \rangle$

(ii) $\langle -1, 1, 1, 1, 1, 1, \ldots \rangle$

4 Attempt any two of the following :

(a) Prove that there are infinitely many positive integers n such that $n^2 + (n+1)^2$ is a perfect square. **7**

(b) Prove that there are infinitely many positive integers n such that $1 + 2 + 3 + \ldots + n = m^2$ for some integer m. **7**

(c) If $\frac{h_j}{k_j}$ denote the j^{th} convergent of an irrational number θ then prove that for all $n \geq 1$: **7**

(i) $\left|\theta k_n - h_n\right| < \left|\theta k_{n-1} - h_{n-1}\right|$

(ii) $\left|\theta - \frac{h_n}{k_n}\right| < \left|\theta - \frac{h_{n-1}}{k_{n-1}}\right|$

5 Do as directed : (Each question carries two marks) **14**

(a) Give the definition of a purely periodic continued fraction.

(b) Express $\frac{2017}{101}$ as a simple continued fraction.

(c) Write down the values of $\frac{h_0}{k_0}, \frac{h_1}{k_1}$ for the continued fraction $\langle 1, 2, 1, 2, \ldots\ldots \rangle$.

(d) Write down all the Farey fractions between 0 and 1 in the rows up to 7^{th} row.

(e) Find first four positive solutions of $x^2 - 2y^2 = 1$.

(f) Find three primitive Pythagorean tripletes (x, y, z) for which $z > 40$.

(g) Find three positive integers n for which $1 + 2 + 3 + \ldots\ldots\ldots\ldots + n$ is a perfect square.